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Contribution to Axisymmetric Flow through Turbomachines

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Introduction

An analysis has been made of incompressible axisymmetric flow in turbomachines. The flowfield is developed for a prescribed velocity distribution along one arbitrary streamline. Each variable is expressed as a series in therms of the stream function and fourth order terms are retained to give an accurate solution. The equations of blade surfaces are deduced. For the case of constant intake energy they are reduced to a partial differential equation which can be easily solved as an ordinary differential equation. The flowfield is illustrated in an example.

Development of the Flow Equations

Euler's equations in Lamb's form for the steady flow in terms of axisymmetric coordinates are

$$F_r = \frac{1}{\varrho}\frac{\partial p}{\partial r} + \frac{\partial}{\partial r}\left(\frac{c^2}{2}\right) - \frac{c_u}{r}\frac{\partial}{\partial r}(r\,c_u)\,, \tag{1}$$

$$F_z = \frac{1}{\varrho}\frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left(\frac{c^2}{2}\right) - \frac{c_u}{r}\frac{\partial}{\partial z}(r\,c_u)\,, \tag{2}$$

$$F_u = \frac{c_r}{r}\frac{\partial}{\partial r}(r\,c_u) + \frac{c_z}{r}\frac{\partial}{\partial z}(r\,c_u)\,, \tag{3}$$

where r and z are radial and axial coordinates, c_r and c_z velocity components in the r- and z-directions respectively, c_u tangential velocity, ϱ density, and F_r, F_u, and F_z are forces acting on the particle of fluid. In steady axisymmetric flow, the continuity relationship is the necessary condition for the existence of a stream function ψ and the assumption of irrotational flow in meridian planes insures that a velocity potential φ exists. Let φ and ψ be the independent variables and the Jacobian of the transformation is not equal to zero in the region of interest. The continuity equation, the condition of irrotational flow, and Equations (1), (2) and (3) may be written as

$$\frac{\partial z}{\partial \varphi} = \frac{\cos\delta}{c_m}\,, \qquad \frac{\partial z}{\partial \psi} = -\frac{\psi\sin\delta}{r\,c_m}\,, \tag{4}$$

$$\frac{\partial r}{\partial \varphi} = \frac{\sin\delta}{c_m}\,, \qquad \frac{\partial r}{\partial \psi} = \frac{\psi\cos\delta}{r\,c_m}\,, \tag{5}$$

$$F_\varphi = \frac{c_m}{\varrho}\frac{\partial p}{\partial \varphi} + c_m\frac{\partial}{\partial \varphi}\left(\frac{c^2}{2}\right) - \frac{c_m\,c_u}{r}\frac{\partial}{\partial \varphi}(r\,c_u)\,, \tag{6}$$

$$F\psi = \frac{r\,c_m}{\varrho\psi}\frac{\partial p}{\partial \psi} + \frac{r\,c_m}{\psi}\frac{\partial}{\partial \psi}\left(\frac{c^2}{2}\right) - \frac{c_m\,c_u}{r}\frac{\partial}{\partial \psi}(r\,c_u)\,, \tag{7}$$

$$F_u = \frac{c_m^2}{r}\frac{\partial}{\partial \varphi}(r\,c_u)\,, \tag{8}$$

where φ and ψ are the coordinates, c_m meridional velocity and δ its direction (the angle between the velocities c_m and c_z). It is desirable that these coordinates have units of length, and therefore new coordinates ξ and η are introduced, which are defined as follows [1]

$$\frac{d\varphi}{d\xi} = c_{m\,R}(\xi)\,, \qquad \psi = c_{m\,m}^{1/2}\,\eta\,. \tag{9}$$

In these equations $c_{m\,R}$ is the meridional velocity distribution along the reference line $r = F(z)$, and $c_{m\,m}$ = constant whose unit is length/time. By transforming Equations (4)—(8), the form becomes

$$\frac{\partial z}{\partial \eta} = -\frac{c_{m\,m}}{c_m}\frac{\eta}{r}\sin\delta\,, \qquad \frac{\partial r}{\partial \eta} = \frac{c_{m\,m}}{c_m}\frac{\eta}{r}\cos\delta\,, \tag{10}$$

$$\frac{\partial}{\partial \eta}\left(\frac{c_{m\,R}}{c_m}\right) = -\frac{c_{m\,m}}{c_m}\frac{\eta}{r}\frac{\partial \delta}{\partial \xi}\,, \qquad \frac{c_{m\,R}}{c_m}\frac{\partial \delta}{\partial \eta} = \frac{\partial}{\partial \xi}\left(\frac{c_{m\,m}}{c_m}\frac{\eta}{r}\right), \tag{11}$$

$$F_\xi = \frac{c_m}{c_{mR}} \frac{\partial}{\partial \xi}\left(\frac{c^2}{2} + \frac{p}{\varrho}\right) - \frac{c_m}{c_{mR}} \frac{c_u}{r} \frac{\partial}{\partial \xi}(r\, c_u)\,, \tag{12}$$

$$F_\eta = \frac{r\, c_m}{\eta\, c_{mm}} \frac{\partial}{\partial \eta}\left(\frac{c^2}{2} + \frac{p}{\varrho}\right) - \frac{c_m}{c_{mm}} \frac{c_u}{\eta} \frac{\partial}{\partial \eta}(r\, c_u)\,, \tag{13}$$

$$F_u = \frac{c_m^2}{r\, c_{mR}} \frac{\partial}{\partial \xi}(r\, c_u)\,. \tag{14}$$

The equation of the blade surfaces for axisymmetric flow may be written as

$$\mu(\xi, \eta) - K\,\Theta = C_1\,, \tag{15}$$

where $\mu(\xi, \eta)$ is any function, K is a constant, and C_1 = constant defining one blade. The forces of blades are

$$F_\xi = \lambda(\xi, \eta) \frac{c_m}{c_{mR}} \frac{\partial \mu}{\partial \xi}\,, \tag{16}$$

$$F_\eta = \lambda(\xi, \eta) \frac{c_m}{c_{mm}} \frac{r}{\eta} \frac{\partial \mu}{\partial \eta}\,, \tag{17}$$

$$F_u = -\lambda(\xi, \eta) \frac{K}{r}\,, \tag{18}$$

where $\lambda(\xi, \eta)$ is any function of ξ and η.

The energy equation is

$$\frac{c^2}{2} + \frac{p}{\varrho} - \omega\, r\, c_u = \Phi(\eta)\,, \tag{19}$$

where ω = constant showing the angular velocity of the runner, and $\Phi(\eta)$ function of η only.

From Equations (10)—(19), by eliminating the blade forces F_ξ, F_η, F_u and pressure p, and introducing the dimensionless variables

$$\bar r = r/R\,, \quad \bar z = z/R\,, \quad \bar\eta = \eta/R\,, \quad \bar\xi = \xi/R\,, \quad \bar f = r\, c_u/R\, c_{mm}\,, \quad \bar\omega = R\,\omega/c_{mm}\,,$$

$$\bar\lambda = \lambda/c_{mm}^2\,, \quad \bar\mu = \mu\,, \quad \bar\Phi(\bar\eta) = R\,\Phi/c_{mm}^2\,, \quad H_R^2 = c_{mm}/c_{mR}\,, \quad H^2 = c_{mm}/c_m\,,$$

where R = constant representing any radius, becomes

$$\frac{\partial \bar z}{\partial \bar\eta} = -H^2 \frac{\bar\eta}{\bar r} \sin\delta\,, \qquad \frac{\partial \bar r}{\partial \bar\eta} = H^2 \frac{\bar\eta}{\bar r} \cos\delta\,, \tag{20}$$

$$\frac{\partial}{\partial \bar\eta}\left(\frac{H}{H_R}\right)^2 = -H^2 \frac{\bar\eta}{\bar r} \frac{\partial \delta}{\partial \bar\xi}\,, \qquad \frac{\partial \delta}{\partial \bar\eta} = \left(\frac{H_R}{H}\right)^2 \frac{\partial}{\partial \bar\xi}\left(H^2 \frac{\bar\eta}{\bar r}\right), \tag{21}$$

$$\bar\lambda \frac{\partial \bar\mu}{\partial \bar\xi} = \left(\bar\omega - \frac{\bar f}{\bar r^2}\right) \frac{\partial \bar f}{\partial \bar\xi}\,, \tag{22}$$

$$\bar\lambda \frac{\partial \bar\mu}{\partial \bar\eta} = \left(\bar\omega - \frac{\bar f}{\bar r^2}\right) \frac{\partial \bar f}{\partial \bar\eta} + \bar\Phi'(\bar\eta)\,, \tag{23}$$

$$-\bar\lambda\, \bar K\, H_R^2 = \left(\frac{H_R}{H}\right)^4 \frac{\partial \bar f}{\partial \bar\xi}\,. \tag{24}$$

To obtain a solution of the differential equations (20) and (21) each of the unknows will be written in terms of series

$$\bar r = F(\bar\xi) + r_1\, \bar\eta + r_2\, \bar\eta^2 + r_3\, \bar\eta^3 + r_4\, \bar\eta^4\,, \tag{25}$$

$$\bar z = \bar\xi + z_1\, \bar\eta + z_2\, \bar\eta^2 + z_3\, \bar\eta^3 + z_4\, \bar\eta^4\,, \tag{26}$$

$$\delta = F'(\bar\xi) + \delta_1\, \bar\eta + \delta_2\, \bar\eta^2 + \delta_3\, \bar\eta^3 + \delta_4\, \bar\eta^4\,, \tag{27}$$

$$(H_R/H)^2 = 1 + c_1\, \bar\eta + c_2\, \bar\eta^2 + c_3\, \bar\eta^3 + c_4\, \bar\eta^4\,. \tag{28}$$

The omission of higher order terms implies that the flow variables can be described by fourth order polynoms. This is based on the physical situation, and justification can be obtained by comparison with experimental results.

By equating the coefficients of like powers of $\bar\eta$, the following are found:

$$r_1 = z_1 = \delta_1 = c_1 = r_3 = z_3 = \delta_3 = c_3 = 0\,, \tag{29}$$

$$r_2 = \frac{H_R^2}{2\,F}\left(1 - \frac{F'^2}{2} + \frac{F'^4}{24}\right), \tag{30}$$

The initial boundary conditions are specified on the reference streamline $\eta = 0$ with $\bar{z} = \bar{\xi}$, $\bar{r} = F(\bar{z})$, $H = H_R(\bar{z})$ and $\delta = F'(\bar{z})$. To obtain the solutions of other variables, Equations (22), (23) and (24) are to be solved. For the case of constant intake energy $\Phi'(\eta) = 0$, the system of equations is reduced to

$$\frac{\partial\mu}{\partial\xi}\frac{\partial\bar{f}}{\partial\eta} = \frac{\partial f}{\partial\xi}\frac{\partial\mu}{\partial\bar{\eta}}, \tag{31}$$

$$\frac{1}{K\,H_R^2}\left(\frac{H_R}{H}\right)^4\frac{\partial\mu}{\partial\xi} + \omega - \frac{f}{r^2} = 0\,. \tag{32}$$

Equation (31) is satisfied when $\bar{f}$ is any function of μ only, $\bar{f} = \bar{f}(\mu)$. Then the system of equations is reduced to Equation (32), which can be integrated as an ordinary differential equation.

Method of Solution

The dimensionless equations (25)—(28) are expressed as functions of H_R, F, and their derivatives with respect to $\bar{\xi}$. For the given boundary conditions, H_R and F are prescribed. These functions are represented by polynoms with coefficients determined by the square root method. Streamlines are given by constant values of $\bar{\eta}$. One is free to select a particular value for $\bar{\eta}$ and to solve the equations for $\bar{r}$ and $\bar{z}$, using the values of the coefficients previously determined. This varying the $\bar{\xi}$ values gives, the streamline for which $\bar{\eta}$ = constant. This method has been programmed for an electronic digital computer.

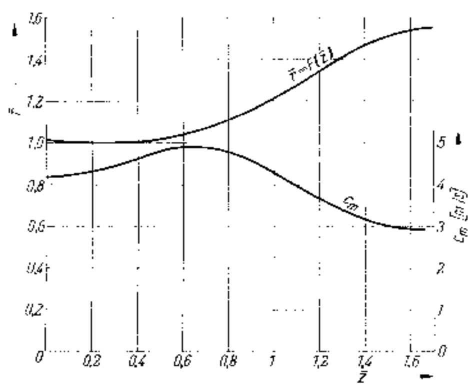

Fig. 1. Boundary conditions

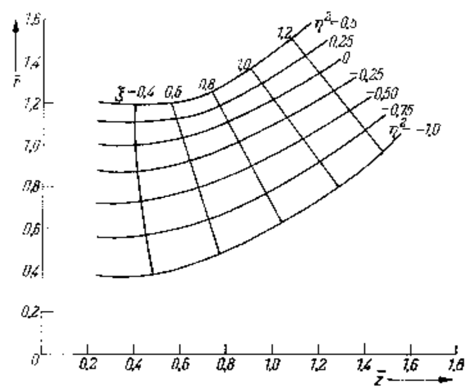

Fig. 2. Flowfield

Results

The flowfield has been computed for the prescribed functions as given in Fig. 1. The results of calculation are shown in Figs. 2 and 3. An ELLIOTT 803 computer in the Computer Center of "Energoprojekt", Belgrade, has been used for all calculations.

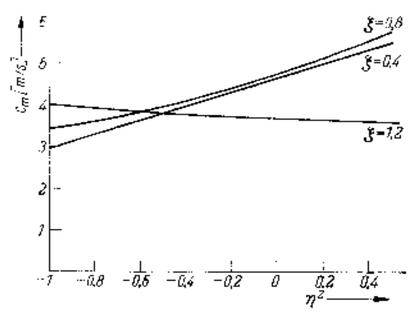

Fig. 3. Velocity distribution

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